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ORIGINAL RESEARCH

A new class of polynomial functions equipped with a parameter

Saeid Abbasbandy[1]

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Abstract In this study, a new class of polynomial functions although equipped with a parameter is introduced. This class can be employed for computational solution of linear or non-linear functional equations, including ordinary differential equations or integral equations. The extra parameter permits us to obtain more accurate results. In the present paper, a number of numerical examples show the ability of this class of polynomial functions.

Keywords Lucas polynomials · Bessel polynomials · Boubaker polynomials · Collocation method

Introduction

In numerous cases, we can find the solution of the linear or non-linear functional equations by Polynomial Expansion Scheme (PES) or matrix methods (in the sense of collocation scheme) based on famous polynomials such as Taylor, Chebyshev, Berstein, Legendre, Laguerre, Hermite, Bessel, Lucas, and Boubaker [1–3, 10–13].

In many problems, utilizing polynomial functions can assist us in finding reasonable solutions. For example, in physics, engineering, and economics, we can find many computational methods applying polynomial functions. A list of polynomial functions and their applications can be found in [2]. In recent years, a great number of new families of polynomials are introduced, including q-analogue of Hermite polynomials [6], d-orthogonal polynomials by Cheikha and Romdhane [5], unified family of generalized Apostol-Bernoulli, and Euler and Genocchi polynomials by El-Desouky and Gomaa [7].

In numerical solution of an Ordinary Differential Equation (ODE), we usually put a linear combination of polynomial functions with unknown coefficients in the main equation and then find the unknowns. This can be done using collocation or, Galerkin or Tau methods. As we know, in this scheme, we can select a set of orthogonal or orthonormal polynomial functions for the solution.

In the current research, we aim to introduce a new class of polynomial functions equipped with a parameter that can be used for computational solution of linear or non-linear functional equations, including ordinary differential equations, partial differential equations, or integral equations. The extra parameter contributes to obtaining more accurate results. Numerical examples demonstrate the ability of this class of polynomial functions to obtain better solutions.

Polynomial functions

In this section, a new class of polynomial functions is introduced. This definition is based on the Chebyshev polynomial of the second kind, $U_n(x)$. These functions have already been employed. For example, Boubaker polynomials are defined as $B_n(x) = U_n(\frac{x}{2}) + 3U_{n-2}(\frac{x}{2})$, for $n \geq 2$. Furthermore, the modified Boubaker polynomials [13] are defined as $\tilde{B}_n(x) = 6xU_{n-1}(x) - 2U_n(x)$, for $n \geq 1$.

Definition 2.1 Suppose that a is an arbitrary constant and $U_n(x)$ is the Chebyshev polynomial of the second kind. Let

✉ Saeid Abbasbandy
abbasbandy@yahoo.com

[1] Department of Mathematics, Imam Khomeini International University, Qazvin 34149-16818, Iran

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$A_0(x) = 1$ for any $n \geq 1$. The new class of polynomial functions is defined by the following:

$$A_n(x) = axU_{n-1}(x) + U_n(x).$$

We can find very easily that

$$\begin{aligned}
A_1(x) =& (2+a)x,\\
A_2(x) =& -1 + 2(2+a)x^2,\\
A_3(x) =& -(4+a)x + 4(2+a)x^3,\\
A_4(x) =& 1 - 4(3+a)x^2 + 8(2+a)x^4,\\
A_5(x) =& (6+a)x - 4(8+3a)x^3 + 16(2+a)x^5,\\
A_6(x) =& -1 + 6(4+a)x^2 - 16(5+2a)x^4 + 32(2+a)x^6,\\
&\vdots\\
A_{n+1}(x) =& 2xA_n(x) - A_{n-1}(x).
\end{aligned}$$

We can see that for $n \geq 2$, $A_n(x) = (1 + \frac{a}{2})U_n(x) + \frac{a}{2}U_{n-2}(x)$.

Proposition 2.1 Let $T_n(x)$ be the Chebyshev polynomial of the first kind. Then, $A_n(x) = (a+1)\frac{x}{n}T'_n(x) + T_n(x)$ for $n \geq 1$.

Proposition 2.2 Let $\lfloor . \rfloor$ designate the floor function. Then, for $n \geq 1$,

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{(a+2)n - 2(a+1)k}{2(n-k)} (2x)^{n-2k}.$$

Proposition 2.3 The polynomial $A_n(x)$ for $n \geq 0$ satisfies the following equation:

$$\begin{aligned}
&(x^2-1)\big(1 + a + n + a(2+a)nx^2\big)y''(x)\\
&+ x\big(3(1+n) + a(3 + (2+a)n(2+x^2))\big)y'(x)\\
&- n\big((1+a+n)(2+2a+n) + a(2+a)n^2x^2\big)y(x) = 0.
\end{aligned}$$

Proposition 2.4 For $n \geq 0$, the following equations are obtained:

$$\sum_{k=0}^{n} A_k(x)A_k(y) = -\frac{a}{2} + \frac{A_{n+1}(x)A_n(y) - A_n(x)A_{n+1}(y)}{2(x-y)},$$

$$\sum_{k=0}^{n} A_k^2(x) = -\frac{a}{2} + \frac{1}{2}\Big(A'_{n+1}(x)A_n(x) - A'_n(x)A_{n+1}(x)\Big).$$

Proposition 2.5 By considering the inner product

$$(f, g) = \int_{-1}^{1} \sqrt{1-x^2} f(x)g(x)\mathrm{d}x,$$

we have

$$(A_n(x), A_m(x)) = \begin{cases} \frac{\pi}{2} & n = m = 0,\\ \frac{1}{8}(2+a)^2\pi & n = m = 1,\\ \frac{1}{4}(2+2a+a^2)\pi & n = m \geq 2,\\ \frac{a\pi}{4} & |n-m| = 2, nm = 0,\\ \frac{1}{8}a(2+a)\pi & |n-m| = 2, nm \neq 0,\\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.1 The system $\{A_{4n}(x)\}_{n=0}^{\infty}$ is an orthogonal system of polynomial functions, with respect to the inner product defined in Proposition 2.5.

The *m*th-order non-linear differential equation

In this section, the new class of polynomial functions is considered to find the numerical solutions of some non-linear differential equations. The solutions obtained show the efficiency of this class of functions. In this class, we have an unknown parameter, and by selecting it, we can reduce the error of PES. In all of the following examples, the least square method is applied so as to find the unknown parameter. All iterative algorithms, which we needed for our initial guess, were initialized at zero.

The *m*th-order non-linear differential equation

$$\sum_{k=0}^{m}\sum_{r=0}^{n} P_{k,r}(x)y^r(x)y^{(k)}(x) = g(x), \quad 0 \leq a \leq x \leq b, \tag{1}$$

with the mixed boundary conditions

$$\sum_{k=0}^{m-1} \left(a_{j,k}y^{(k)}(a) + b_{j,k}y^{(k)}\right) = \gamma_j, \quad j = 0, 1, \ldots, m-1, \tag{2}$$

where y is an unknown function, g and $P_{k,r}$ are continuous functions, and $a_{j,k}$, $b_{j,k}$, and γ_j are real or complex constants, is considered in [13]. By PFS collocation method, we can solve this problem. In this method, we approximate the solution of (1) in the truncated series form:

$$y(x) \approx y_N(x) = \sum_{k=0}^{N} c_k A_k(x), \tag{3}$$

where c_k, $k = 0, 1, \ldots, N$ are the unknown coefficients. In this method, $N \geq m$ is selected in any positive integer number. The matrix form of the collocation method is

Table 1 Numerical results of the absolute error functions of Example (3.1), with $a = 2.59490427$

x_i	$e_4(x_i)$, [4]	$e_5(x_i)$, [9]	$e_5(x_i)$, [13]	$e_5(x_i)$
0	0	0	0	0
0.2	5.81371970e−3	1.52386520e4	3.10830422e−4	3.10830419e−4
0.4	8.16606722e−2	1.1431338878e2	8.84574370e−5	8.84574342e−5
0.6	3.73690421e−1	1.2118642056e−1	1.61217488e−4	1.61217491e−4
0.8	1.08914877e−0	6.1674343718e−1	1.07446634e−4	1.07446646e−4
1.0	2.48769984e−0	2.1293665035e−0	3.60206386e−3	3.60206385e−4

Table 2 Numerical results of the absolute error functions of Example (3.1), with $a = 13.50332999$

x_i	Boubaker functions $e_{10}(x_i)$	Present method $e_{10}(x_i)$
0	0	0
0.2	3.2921789994145e−7	3.2921717829648e−7
0.4	2.2017646683636e−7	2.2017604695002e−7
0.6	1.4863375341623e−7	1.4863314123925e−7
0.8	7.0191910173101e−8	7.0190931511505e−8
1.0	1.1546738372115e−5	1.1546784083105e−5

discussed in [13]. In this method, a non-linear system with $N+1$ equations is constructed to find $N+1$ unknowns. However, here, we have $N+2$ unknowns, i.e., c_k, $k = 0, 1, \ldots, N$ and the unknown parameter of a, of the new class of polynomial functions of A_k. By adding one collocation point to the set of points, we can construct a non-linear system with $N+2$ equations. Via the least square method, we can minimize the L_2 norm of the residual in the augmented non-linear system to obtain a. Afterwards, we can continue the matrix collocation method to find other unknowns like [13].

Example 3.1 We consider the Riccati differential equation:

$$y'(x) = y(x) - 2y^2(x), \quad 0 \le x \le 1,$$

with the initial condition $y(0) = 1$, [4, 9, 13], which is a special case of (1). The exact solution is $y(x) = 1/(2 - e^{-x})$. Table 1 indicates the absolute error $e_N(x) = |y(x) - y_N(x)|$ for different methods, including the Taylor method [4], the decomposition method [9], the PES by Bessel polynomials [13], and the present method. Table 2 demonstrates a comparison between the new class of polynomial functions in present method and Boubaker polynomials, pointing to the efficiency of the new class of polynomial functions.

Example 3.2 We consider the non-linear Abel differential equation of the second kind:

$$y(x)y'(x) + xy(x) + y^2(x) + x^2y^3(x) = xe^{-x} + x^2e^{-3x}, \quad 0 \le x \le 1,$$

with the initial condition $y(0) = 1$, [8, 13], which is special case of (1). The exact solution is $y(x) = e^{-x}$. Table 3 shows the absolute error $e_N(x)$ for different methods. The Taylor method [8] in first column, the PES by Bessel polynomials in second column, the PES by Boubaker polynomials in third column, and the present method in last column are shown. Table 4 shows a comparison with Boubaker polynomials, and so shows the efficiency of the new class of polynomial functions.

Table 3 Numerical results of the absolute error functions of Example (3.2), with $a = 4.01764012$

x_i	Taylor method $e_5(x_i)$, [8]	Bessel functions $e_5(x_i)$	Boubaker functions $e_5(x_i)$	Present method $e_5(x_i)$
0	0	0	0	0
0.2	8.64080e−8	6.8031096090e−7	6.8031096079e−7	6.8031096090e−7
0.4	5.379366e−6	2.2898841512e−7	2.2898841490e−7	2.2898841501e−7
0.6	5.9636094e−5	4.0321424399e−7	4.0321424366e−7	4.0321424377e−7
0.8	3.26297447e−4	4.4156961293e−7	4.4156961343e−7	4.4156961315e−7
1.0	1.212774501e−3	1.4263286683e−5	1.4263286683e−5	1.4263286682e−5

Table 4 Numerical results of the absolute error functions of Example (3.1), with $a = 37.74361705$

x_i	Boubaker functions $e_{10}(x_i)$	Present method $e_{10}(x_i)$
0	0	0
0.2	2.8976820942717e−14	2.8754776337792e−14
0.4	2.1649348980191e−14	2.1649348980191e−14
0.6	1.3322676295502e−14	1.3211653993039e−14
0.8	2.6090241078691e−15	2.0539125955565e−15
1.0	1.8128276657592e−12	1.8171020244040e−12

Conclusion

In this study, a new class of polynomial functions equipped with a parameter was introduced. We found that the extra parameter permits us to obtain more accurate results. By solving Riccati and Abel non-linear differential equations, the ability and efficiency of this class of polynomial functions were supported. The application of this class of polynomials in solving complicated ordinary differential equations and partial differential equations and integral equations is expected.

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